

ZEROS OF DIRICHLET POLYNOMIALS VIA A DENSITY CRITERION

WILLIAN D. OLIVEIRA

ABSTRACT. We obtain a necessary and sufficient condition in order that a semi-plane of the form $\Re(s) > r$, $r \in \mathbb{R}$, is free of zeros of a given Dirichlet polynomial. The result may be considered a natural generalization of a well-known criterion for the truth of the Riemann hypothesis due to Báez-Duarte. An analog for the case of Dirichlet polynomials of a result of Burnol which is closely related to Báez-Duarte's one is also established.

1. INTRODUCTION AND STATEMENT OF RESULTS

A series of recent results were devoted to the connection between zero-free regions of some Dirichlet series and density of certain spaces of functions in $L^2(0, \infty)$. The majority of the papers concern the Riemann zeta function because of its importance in the study of the distribution of primes.

The space \mathcal{B} of Beurling functions is defined by

$$(1.1) \quad \mathcal{B} = \left\{ h : (0, 1) \rightarrow \mathbb{C} \mid h(x) = \sum_{k=1}^n b_k \left\{ \frac{1}{\theta_k x} \right\}, \quad b_k \in \mathbb{C}, \quad \theta_k \geq 1, \quad \sum b_k / \theta_k = 0 \right\},$$

where $\{x\} = x - [x]$ stands for the fractional part of x . If \mathcal{B}^p denotes the closure of \mathcal{B} in $L^p(0, 1)$, the following classical result holds:

Theorem A. *The Riemann zeta function $\zeta(s)$ does not vanish in the semi-plane $\Re(s) > 1/p$ if and only if $\mathcal{B}^p = L^p(0, 1)$.*

Nyman [22] proved the result for $p = 2$ in 1950 and Beurling [?] generalised it for $p > 1$ in 1955. That is why nowadays the statement of the theorem is commonly known as the Nyman-Beurling criterion. Bercovici and Foias [8] settled the case $p = 1$ in 1984. It is clear that it does not hold for $p > 2$. Since then various results related to Theorem A have been obtained [1, 3, 4, 5, 6, 9, 12, 13, 14, 15, 17, 18, ?]. For $p = 2$ the above theorem provides a criterion for the truth of the Riemann hypothesis (RH).

Given a real number $\lambda \geq 1$, consider the subspace $\mathcal{B}_\lambda \subset \mathcal{B}$ which consists of the functions $f \in \mathcal{B}$, such that $\theta_k \leq \lambda$ for every $k = 1, \dots, n$, and denote by $D_{\lambda,p}$ the distance in $L^p(0, 1)$ from the characteristic function $\mathbf{1}_{(0,1)}$ of the interval $(0, 1)$ to the space \mathcal{B}_λ . Since Beurling [?] also showed that $\mathcal{B}^p = L^p(0, 1)$ if and only if $\mathbf{1}_{(0,1)} \in \mathcal{B}^p$, then the above theorem can be reformulated as

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Theorem A *The Riemann zeta function $\zeta(s)$ does not vanish in the semi-plane $\Re(s) > 1/p$ if and only if*

$$\lim_{\lambda \rightarrow \infty} D_{\lambda,p} = 0.$$

Báez-Duarte [2] showed that when $p = 2$ the conditions $\theta_k \geq 1$ in (1.1) can be reduced to $\theta_k \in \mathbb{N}$ and, if we substitute $L^2(0, 1)$ by $L^2(0, \infty)$, the restriction $\sum b_k/\theta_k = 0$ can be removed. Báez-Duarte's refinements yield the following nice criterion for the Riemann hypothesis in terms of approximation of the characteristic function $\mathbf{1}_{(0,1)}$:

Theorem B. *The RH holds if and only if $\lim_{n \rightarrow \infty} d_n = 0$, where*

$$d_n^2 = \inf_{b_1, \dots, b_n \in \mathbb{C}} \int_0^\infty \left| \mathbf{1}_{(0,1)} - \sum_{k=1}^n b_k \left\{ \frac{1}{kx} \right\} \right|^2 dx.$$

The latter statement reduces the RH to an extremal problem about the best approximation of $\mathbf{1}_{(0,1)}$ in a Hilbert space in terms of elements from a finite dimensional subspace and the solution of every such a problem is given by the projection. As it is clear from Theorem B, d_n is nothing but the distance in $L^2(0, \infty)$ from $\mathbf{1}_{(0,1)}$ to the n -dimensional space $\text{span}\{\rho_k(x) : k = 1, \dots, n\}$, where $\rho_k(x) = \{1/kx\}$, so that

$$d_n^2 = \frac{\det G(\rho_1, \dots, \rho_n, \mathbf{1}_{(0,1)})}{\det G(\rho_1, \dots, \rho_n)},$$

where $G(\rho_1, \dots, \rho_n, \mathbf{1}_{(0,1)})$ and $G(\rho_1, \dots, \rho_n)$ are the Gram matrices of the corresponding functions with respect to the usual inner product in $L^2(0, \infty)$.

It is known that if d_n converges to zero, the speed of convergence would be relatively slow. More precisely, Burnol [12], generalizing an earlier result of Báez-Duarte, Balazard, Landreau and Saias [4] proved that

Theorem C. *The sequence d_n satisfies*

$$\liminf_{n \rightarrow \infty} d_n^2 \log n \geq C,$$

with

$$C = \sum_{\Re(\rho)=1/2} \frac{m_\rho^2}{|\rho|^2},$$

where the sum is over the distinct nontrivial zeros ρ of the Riemann zeta function and m_ρ are their multiplicities.

The Nyman-Beurling criterion (Theorem A) was generalized recently for other general Dirichlet series, including Dirichlet polynomials (DP) [15, 13, 14]. In particular, given a Dirichlet polynomial $P(s) = \sum_{k=1}^m a_k k^{-s}$ of order m , with $a_1 \neq 0$, and $r \in \mathbb{R}$, define the function $\kappa_r(x) : (0, \infty) \mapsto \mathbb{C}$ by

$$\kappa_r(x) = \sum_{k \leq x} \frac{a_k}{k^{r-1/2}}$$

and the space B_r by

$$B_r := \left\{ h : (0, 1) \mapsto \mathbb{C} \mid h(x) = \sum_{k=1}^n b_k \kappa_r \left(\frac{1}{\theta_k x} \right), b_k \in \mathbb{C}, \theta_k \geq 1 \right\}.$$

By analogy with Theorem A, for every $\lambda \geq 1$, fix the subspace $B_{\lambda,r}$ of B_r to consist of those functions $f \in B_r$ that satisfy $\theta_k \leq \lambda$ for every $k = 1, \dots, n$, and denote by $D_{\lambda,r}$ the distance in $L^2(0, 1)$ from $\mathbf{1}_{(0,1)}$ to $B_{\lambda,r}$. The results obtained in [15, 13, 14] imply:

Theorem D. *The Dirichlet polynomial $P(s)$ does not vanish for $\Re(s) > r$ if and only if*

$$\lim_{\lambda \rightarrow \infty} D_{\lambda, r} = 0.$$

Observe that the latter is the natural analog of the Nyman-Beurling criterion for DP. However, to the best of our knowledge, there are no results in the literature which may be considered analogs of Theorems B and C about DP. The main aim of this paper is to fill this gap. Our main results read as follows:

Theorem 1. *The Dirichlet polynomial $P(s) = \sum_{k=1}^m a_k k^{-s}$, with $a_1 \neq 0$, does not vanish in the semi-plane $\Re(s) > r$ if and only if $\lim_{n \rightarrow \infty} d_{n, r} = 0$, where*

$$d_{n, r}^2 = \inf_{b_1, \dots, b_n \in \mathbb{C}} \int_0^\infty \left| \mathbf{1}_{(0,1)} - \sum_{k=1}^n b_k \kappa_r \left(\frac{1}{kx} \right) \right|^2 dx.$$

and

Theorem 2. *For the Dirichlet polynomial $P(s) = \sum_{k=1}^m a_k k^{-s}$, $a_1 \neq 0$, the associated sequence $d_{n, r}$ satisfies*

$$\liminf_{n \rightarrow \infty} d_{n, r}^2 \log n \geq C,$$

with

$$C = \sum_{\Re(\rho_P)=r} \frac{1}{|\rho_P - r + 1/2|^2},$$

where the sum is over the zeros ρ_P of $P(s)$ which belong to the line $\Re(s) = r$. If P does not possess zeros on $\Re(s) = r$ the constant C is defined to be zero.

It is worth mentioning that de Roton [15, 16] obtained nice generalizations of Theorems B and C for zeros of L -functions in the Selberg class. The methods employed in [15, 16] require deep analysis and a crucial fact which allows the generalizations is that those functions obey a functional equation which is a natural analog of the corresponding one for the Riemann zeta function itself. However, in general the Dirichlet polynomials do not obey such structural properties, so that the proofs of our main results require different techniques. Another feature of Theorems 1 is that our criterion concerns lack of zeros in the semi-planes $\Re(s) > r$, for every $r \in \mathbb{R}$, while both Baez-Duarte's criterion and de Roton's generalization deal exclusively with the semi-plane $\Re(s) > 1/2$. From this point of view Theorem 1 in the present paper is more similar to the result obtained in [14] where Baez-Duarte's result was extended for a class of L -functions which is a bit narrower than in de Roton's result but for zeros in the semi-plane of the form $\Re(s) > 1/p$ for any $p \in (1, 2]$.

2. PRELIMINARY RESULTS

In this section we furnish the necessary definitions and known results for the proofs of the main theorems.

2.1. Definitions and basic properties of DP. Let $P(s) = \sum_{k=1}^m a_k k^{-s}$ be a DP of order m with $a_1 \neq 0$. Recall that we associate with P the function $\kappa_r(x) : \mathbb{R}_+ \mapsto \mathbb{C}$, where $\mathbb{R}_+ = (0, \infty)$, defined by

$$(2.1) \quad \kappa_r(x) = \sum_{k \leq x} \frac{a_k}{k^{r-1/2}}$$

and the space

$$(2.2) \quad C_r := \left\{ h : (0, 1) \mapsto \mathbb{C} \mid h(x) = \sum_{k=1}^n b_k \kappa_r \left(\frac{1}{kx} \right), \ b_k \in \mathbb{C} \right\}.$$

The inequality

$$|P(s) - a_1| \leq \sum_{k=2}^m \frac{|a_k|}{k^{\Re(s)}}$$

shows that $P(s) \rightarrow a_1$ uniformly with respect to $\Im(t)$ when $\Re(s) \rightarrow \infty$. On the other hand, $P(s)m^s \rightarrow a_m$ also uniformly with respect to $\Im(t)$ when $\Re(s) \rightarrow -\infty$. Therefore, since a_1 and a_m are nonzero, there are real constants α and β associated with P such that

$$(2.3) \quad \alpha \leq \Re(\rho_P) \leq \beta$$

for every zero ρ_P of P .

Since $P(s)$ is a finite sum of exponential functions, it is an entire function of order one and finite type. Then, by the Hadamard factorization theorem, there are complex constants a, b and a natural number q such that

$$P(s) = s^q e^{a+bs} \prod_{\rho_P} \left(1 - \frac{s}{\rho_P} \right) e^{s/\rho_P},$$

where ρ_P runs over the zeros of P arranged in increasing order of $|\rho_P|$ and $\sum_{\rho_P} |\rho_P|^{-2} < \infty$. Moreover the infinite product converges uniformly on the compact sets of \mathbb{C} .

Given a Dirichlet series $f(s) = \sum_{k=1}^{\infty} a_k k^{-s}$, with $a_1 \neq 0$, we denote by $\sigma_a = \sigma_a(f)$ its abscissa of absolute convergence and we set $\sum_{k=1}^{\infty} \mu_f(k) k^{-s}$ to be the formal Dirichlet series associated with $1/f(s)$.

2.2. The Mellin Transform. The Mellin transform of a function $f : \mathbb{R}_+ \mapsto \mathbb{C}$ is the complex-valued function

$$(2.4) \quad \mathcal{M}[f(x); s] = \mathcal{M}f(s) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} f(x) x^{s-1} dx,$$

defined for those complex numbers s for which the integral converges and the Fourier transform \mathcal{F} of a function $f \in L^1(\mathbb{R})$ is defined by

$$\mathcal{F}[f(x); t] = \mathcal{F}f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixt} dx.$$

Note that the terms Mellin transform and Fourier transform are also applied to the mapping which takes f to $\mathcal{M}f$ and f to $\mathcal{F}f$. As it is known, the Plancherel theorem allows the Fourier transform to be extended to an isometry $\mathcal{F} : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$, called the extended Fourier transform or the Fourier-Plancherel transform. Consider the isometry $\mathcal{I} : L^2(\mathbb{R}_+) \mapsto L^2(\mathbb{R})$ which associates, with each $f \in L^2(\mathbb{R}_+)$, the function $g(u) = f(e^{-u})e^{-u/2}$ in $L^2(\mathbb{R})$. Then the extended Fourier transform allows us to define the extended Mellin transform $\mathcal{M} : L^2(\mathbb{R}_+) \mapsto L^2(\Re(s) = 1/2)$ by

$$\mathcal{M}[f(x); s] = \mathcal{F}[\mathcal{I}f(x); t], \quad s = 1/2 + it.$$

When $\mathcal{I}f \in L^1(\mathbb{R})$, for some $f \in L^2(\mathbb{R}_+)$, the Mellin transform $\mathcal{M}[f(x); 1/2 + it]$ is given by (2.4).

In the proof of Theorem 1 we shall need the Mellin transform of the functions in C_r . It follows from Abel's identity that

$$\sum_{k \leq n} \frac{a_k}{k^{r-1/2}} \frac{1}{k^s} = \frac{\kappa_r(n)}{n^s} + s \int_1^n \frac{\kappa_r(y)}{y^{s+1}} dy.$$

Since κ_r is bounded in \mathbb{R}_+ , if $\Re(s) > 0$, we let $n \rightarrow \infty$ in the latter to obtain

$$\frac{P(s+r-1/2)}{s} = \int_1^\infty \frac{\kappa_r(y)}{y^{s+1}} dy.$$

If $k \in \mathbb{N}$ the change of variables $y = 1/(kx)$ yields

$$\frac{P(s+r-1/2)}{s} k^{-s} = \int_0^{1/k} \kappa_r(1/kx) x^{s-1} dx = \int_0^1 \kappa_r(1/kx) x^{s-1} dx.$$

Therefore, for every $h(x) = \sum_{k=1}^n b_k \kappa_r(1/kx)$ in C_r , we obtain

$$(2.5) \quad \int_0^1 h(x) x^{s-1} dx = \frac{P(s+r-1/2)}{s} \sum_{k=1}^n \frac{b_k}{k^s}, \quad \Re(s) > 0.$$

2.3. Some technical results. We shall need the first effective Perron formula (see [23, Chapter II.2, p. 132]):

Lemma A. *Let $F(w) = \sum a_k k^{-w}$ be a Dirichlet series. Then, for every $c > \max\{0, \sigma_a\}$, $T \geq 1$, and $x \geq 1$ not an integer, we have*

$$\sum_{k < x} a_k = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(w) \frac{x^w}{w} dw + O\left(x^c \sum_{k=1}^\infty \frac{|a_k|}{k^c(1+T|\log(x/k)|)}\right).$$

The following is Lemma 6.30 in [7]:

Lemma B. *Let $f(s) = \sum a_k k^{-s}$ be a Dirichlet series with $a_1 \neq 0$. The Dirichlet series $f(s)$ has finite abscissa of convergence if and only if $f^{-1}(s) = \sum \mu_f(k) k^{-s}$ has finite abscissa of convergence.*

Next we prove two technical lemmas.

Lemma 1. *Let $f(s) = \sum a_k k^{-s}$ be a Dirichlet series with $a_1 \neq 0$ and abscissa of absolute convergence σ_a . If $f(s)$ does not vanish in $\Re(s) > \sigma_z \geq \sigma_a$ then for each $\epsilon > 0$*

$$|f(s)|^{\pm 1} \leq C_\epsilon,$$

uniformly in $\Re(s) \geq \sigma_z + \epsilon$.

Proof. Let $b > \sigma_z + \epsilon$. Since the Dirichlet series $f(s)$ converges absolutely for $\Re(s) > \sigma_z$, there exists a positive constant $C_1(\epsilon)$, such that $\Re(\log f(s)) = \log |f(s)| < C_1(\epsilon)$ for $\Re(s) \geq \sigma_z + \epsilon/2$. Next we apply the Borel-Carathéodory theorem for $\log f(s)$, which is holomorphic for $\sigma > \sigma_z$, and the concentric circumferences with centre at $b + it$ and radii $b - \sigma_z - \epsilon/2$ and $b - \sigma_z - \epsilon$. Then in the smaller circumference

$$(2.6) \quad |\log f(s)| \leq \frac{4b - 4\sigma_z - 4\epsilon}{\epsilon} C_1(\epsilon) + \frac{4b - 4\sigma_z - 3\epsilon}{\epsilon} |\log f(b + it)|.$$

Observe that the straightforward inequality

$$|f(b + it) - a_1| \leq \sum_{k=2}^\infty \frac{|a_k|}{k^b} \leq \frac{1}{2^{b-\sigma_z-\epsilon}} \sum_{k=2}^\infty \frac{|a_k|}{k^{\sigma_z+\epsilon}}$$

and the fact that $a_1 \neq 0$ implies that for a fixed sufficiently large b there exists another positive constant $C_2(\epsilon)$ such that $|\log f(b+it)| < C_2(\epsilon)$ for every real t . Setting the latter inequality into (2.6) we conclude that

$$|\log f(s)| \leq (4b - 4\sigma_z) \frac{\max\{C_1(\epsilon), C_2(\epsilon)\}}{\epsilon} = C(\epsilon, b), \quad \text{for } \sigma_z + \epsilon \leq \Re(s) \leq b.$$

Hence

$$|f(s)|^{\pm 1} \leq e^{C(\epsilon, b)}, \quad \sigma_z + \epsilon \leq \Re(s) \leq b.$$

Applying the Phragmén-Lindelöf convexity principle, as stated in [?, Theorem 5.53], we obtain the desired result. \square

Lemma 2. *Let $f(s) = \sum a_k k^{-s}$ be a Dirichlet series with $a_1 \neq 0$ and abscissa of absolute convergence σ_a . If $f(s)$ does not vanish in $\Re(s) > \sigma_z \geq \sigma_a$ then, for each $\epsilon > 0$ and every $\delta \in (0, \epsilon)$ we have*

$$\sum_{k=1}^n \frac{\mu_f(k)}{k^s} = \frac{1}{f(s)} + O_{\epsilon, \delta}(n^{-\delta})$$

uniformly in the half-plane $\Re(s) \geq \sigma_z + \epsilon$. Moreover,

$$\sum_{k=1}^n \mu_f(k) k^{-s} = O_{\epsilon}(1)$$

uniformly in $n = 1, 2, 3, \dots$ and uniformly in the half-plane $\Re(s) \geq \sigma_z + \epsilon$.

Proof. For s in the half-plane $\Re(s) \geq \sigma_z + \epsilon$, consider the function

$$F(w) = \frac{1}{f(s+w)} = \sum_{k=1}^{\infty} \frac{\mu_f(k)}{k^s} \frac{1}{k^w}.$$

By Lemma B the Dirichlet series $F(w)$ has a finite abscissa of absolute convergence $\sigma_a(F)$. Let us apply Lemma A for fixed $c > \max\{0, \sigma_a(F) + 1\}$ and $x \in 1/2 + \mathbb{N}$ to obtain

$$\sum_{k < x} \frac{\mu_f(k)}{k^s} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(w) \frac{x^w}{w} dw + O\left(x^c \sum_{k=1}^{\infty} \frac{|\mu_f(k)|}{k^{\Re(s)+c}(1+T|\log(x/k)|)}\right).$$

Observe that, for each $k \in \mathbb{N}$, we have $|\log(x/k)| \geq |\log(1+1/2k)| \geq \log 2/(2k)$. Thus

$$O\left(x^c \sum_{k=1}^{\infty} \frac{|\mu_f(k)|}{k^{\Re(s)+c}(1+T|\log(x/k)|)}\right) = O\left(\frac{x^c}{T} \sum_{k=1}^{\infty} \frac{|\mu_f(k)|}{k^{\Re(s)+c-1}}\right) = O\left(\frac{x^c}{T}\right).$$

Deforming the contour of integration along the segment $[c-iT, c+iT]$ into the polygonal line passing through the points $-\delta-iT$ and $-\delta+iT$ and having in mind that the value of the residue of the integrand at the pole $w = 0$ is $1/f(s)$, we obtain

$$\sum_{k < x} \frac{\mu_f(k)}{k^s} = \frac{1}{f(s)} + \frac{1}{2\pi i} \left(\int_{-\delta+iT}^{c+iT} + \int_{-\delta-iT}^{c-iT} + \int_{c-iT}^{-\delta-iT} \right) F(w) \frac{x^w}{w} dw + O\left(\frac{x^c}{T}\right).$$

By Lemma 1 the first and the third integrals can be estimated from above by

$$O_{\epsilon, \delta}\left(\frac{x^c}{T}\right)$$

and the second one by

$$O_{\epsilon, \delta}(x^{-\delta}).$$

Choosing $T = x^{c+\delta}$ we obtain, for each $\epsilon > 0$, every $\delta \in (0, \epsilon)$ and s in the half plane $\Re(s) \geq \sigma_z + \epsilon$,

$$\sum_{k < x} \frac{\mu_f(k)}{k^s} = \frac{1}{f(s)} + O_{\epsilon, \delta}(x^{-\delta}).$$

In particular

$$\sum_{k=1}^n \frac{\mu_f(k)}{k^s} = \frac{1}{f(s)} + O_{\epsilon, \delta}(n^{-\delta})$$

uniformly in $\Re(s) \geq \sigma_z + \epsilon$. Choosing $\delta = \epsilon/2$, by Lemma 1 we obtain

$$\sum_{k=1}^n \mu_f(k) k^{-s} = \frac{1}{f(s)} + O_{\epsilon}(n^{-\epsilon/2}) = O_{\epsilon}(1)$$

uniformly in $n = 1, 2, 3, \dots$ and uniformly in the half plane $\Re(s) \geq \sigma_z + \epsilon$. \square

2.4. Lubinsky's Dirichlet orthogonal polynomials. For any strictly increasing sequence of real numbers $1 = \lambda_1 < \lambda_2 < \lambda_3 < \dots$, Lubinsky [20] considered the general Dirichlet polynomials formed by linear combinations of λ_k^{-it} and obtained the corresponding orthogonal basis with respect to the arctangent density. He proved that the general Dirichlet polynomials $\phi_1(t) = 1$, $\phi_n(t) = (\lambda_n^{1-it} - \lambda_{n-1}^{1-it})/\sqrt{\lambda_n^2 - \lambda_{n-1}^2}$, $n \geq 2$, satisfy

$$\int_{\mathbb{R}} \phi_n(t) \overline{\phi_m(t)} \frac{dt}{\pi(1+t^2)} = \delta_{nm}, \quad n, m \in \mathbb{N}.$$

Choosing $\lambda_n = n^{1/2}$ and performing a simple change of variables one concludes that the Dirichlet polynomials $\psi_n(t) = n^{1/2-it} - (n-1)^{1/2-it}$, $n \geq 2$, obey

$$\frac{1}{2\pi} \int_{\mathbb{R}} \psi_n(t) \overline{\psi_m(t)} \frac{dt}{1/4 + t^2} = \delta_{nm}, \quad n, m \in \mathbb{N}.$$

Moreover Lubinsky [20, (1.20), (1.19)] proved that the kernel polynomials

$$K_n(u, v) = \sum_{k=1}^n \psi_k(u) \overline{\psi_k(v)}$$

satisfy the following uniform asymptotic estimates in compact subsets of \mathbb{R} , as $n \rightarrow \infty$:

$$(2.7) \quad K_n(u, u) = |1/2 + iu|^2 \log n (1 + o(1))$$

and

$$(2.8) \quad |K_n(u, v)| \leq \frac{2|1/2 + iu| |1/2 - iv|}{|u - v|} + o(\log n).$$

The following simple fact holds:

Lemma 3. *For every $l \in \mathbb{N}$ and for any distinct numbers $t_1, \dots, t_l \in \mathbb{R}$, there exists $n(l) \in \mathbb{N}$ such that the self-adjoint matrix $H = (K_n(t_i, t_j))_{i,j=1}^l$ is nonsingular for every $n > n(l)$. Moreover,*

$$(2.9) \quad \frac{\det H}{(\log n)^l} = |1/2 + it_1|^2 \dots |1/2 + it_l|^2 + o(1) \quad \text{as } n \rightarrow \infty.$$

Proof. The Leibniz formula for the expansion of the determinant H in terms of the permutations \mathcal{P}_l and the above asymptotic formulae for the kernel polynomials yield

$$\begin{aligned} \det H &= \sum_{\sigma \in \mathcal{P}_l} \operatorname{sgn}(\sigma) K_{n,r}(t_1, t_{\sigma(1)}) \dots K_{n,r}(t_l, t_{\sigma(l)}) \\ &= |1/2 + it_1|^2 \dots |1/2 + it_l|^2 (\log n)^l (1 + o(1)) + O((\log n)^{l-2}) \\ &= |1/2 + it_1|^2 \dots |1/2 + it_l|^2 (\log n)^l + o((\log n)^l), \end{aligned}$$

which is equivalent to (2.9). Thus, obviously H is nonsingular for all sufficiently large n . \square

3. THE ANALOG OF BÁEZ-DUARTE'S CRITERION FOR DIRICHLET POLYNOMIALS

Proof of Theorem 1. Suppose first that $\lim_{n \rightarrow \infty} d_{n,r} = 0$. This means that $\mathbf{1}_{(0,1)}$ belongs to the closure of the space C_r , defined by (2.2), in $L^2(0, 1)$. Then, given $\epsilon > 0$, there exists $h \in C_r$ such that $\|\mathbf{1}_{(0,1)} - h\|_2 < \epsilon$. By (2.5)

$$\int_0^1 (1 - h(x)) x^{s-1} dx = \frac{1 - P(s + r - 1/2) \sum_{k=1}^n b_k k^{-s}}{s}, \quad \Re(s) > 0.$$

It is clear that $x^{s-1} \in L_2(0, 1)$ provided $\Re(s) > 1/2$. Moreover,

$$\|x^{s-1}\|_2^2 = \frac{1}{2\Re(s) - 1}.$$

Hölder's inequality yields

$$|1 - P(s + r - 1/2) \sum_{k=1}^n b_k k^{-s}|^2 < \epsilon^2 \frac{|s|^2}{2\Re(s) - 1}.$$

Assume that $P(s)$ possesses a zero ρ_P in the semi-plane $\Re(s) > r$. Letting $\epsilon \rightarrow 0$ in the latter inequality we obtain an obvious contradiction. Therefore $P(s)$ does not vanish in $\Re(s) > r$.

Now suppose that $P(s)$ does not vanish in $\Re(s) > r$. In order to show that $d_{n,r}$ converges to zero it suffices to show that the function $\mathbf{1}_{(0,1)}$ belongs to the closure of C_r in $L^2(0, 1)$. However, the latter statement is obviously equivalent to the fact that $\mathcal{M}(\mathbf{1}_{(0,1)})$ belongs to the closure of $\mathcal{M}(C_r)$ in $L^2(\Re(s) = 1/2)$ because the Mellin transform $\mathcal{M} : L^2(\mathbb{R}_+) \mapsto L^2(\Re(s) = 1/2)$ is an isometry. Thus our aim is to prove that the closure of $\mathcal{M}(C_r)$ contains the function $\mathcal{M}(\mathbf{1}_{(0,1)})$.

From now on, up to the end of the present theorem, we set $s = 1/2 + it$. First we observe that, by (2.5),

$$\mathcal{M}(C_r) = \left\{ \frac{P(s + r - 1/2) \sum_{k=1}^n b_k k^{-s}}{\sqrt{2\pi s}} : b_k \in \mathbb{C} \right\}.$$

For each $n \in \mathbb{N}$ and $\epsilon > 0$ we define the function $H_{n,\epsilon} \in L^2(\Re(s) = 1/2)$ by

$$H_{n,\epsilon}(s) = \frac{P(s + r - 1/2)}{\sqrt{2\pi s}} \sum_{k=1}^n \frac{\mu_P(k)}{k^{r+\epsilon-1/2}} \frac{1}{k^s},$$

where $\mu_P(k)$ are the coefficients of the expansion of $1/P$ in a formal Dirichlet series. Clearly $H_{n,\epsilon}$ belongs to $\mathcal{M}(C_r)$. Since by hypothesis the Dirichlet Polynomial $P(s)$

does not vanish when $\Re(s) > r$ and converges absolutely in this semi-plane, for every fixed ϵ we can apply Lemma 2 to obtain

$$\lim_{n \rightarrow \infty} H_{n,\epsilon}(s) = H_\epsilon(s),$$

where

$$H_\epsilon(s) := \frac{1}{\sqrt{2\pi s}} \frac{P(s+r-1/2)}{P(s+r+\epsilon-1/2)}.$$

It follows also from Lemma 2 that for every fixed ϵ the modulus of $H_{n,\epsilon}$ is uniformly bounded with respect to n by a function from $L^2(\Re(s) = 1/2)$. Hence, by Lebesgue's dominated convergence theorem, for every fixed $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} H_{n,\epsilon} = H_\epsilon,$$

in the $L^2(\Re(s) = 1/2)$ norm. Since $H_{n,\epsilon} \in \mathcal{M}(C_r)$ for every fixed $\epsilon > 0$, then H_ϵ belongs to the closure of $\mathcal{M}(C_r)$.

By the definition of H_ϵ ,

$$\lim_{\epsilon \rightarrow 0} H_\epsilon(s) = \frac{1}{\sqrt{2\pi s}} = \mathcal{M}[\mathbf{1}_{(0,1)}(x); s].$$

Next we show that the modulus of H_ϵ is uniformly bounded with respect to ϵ , as $\epsilon \rightarrow 0$, by a function from $L^2(\Re(s) = 1/2)$. Recall the Hadamard factorisation of $P(z)$,

$$P(z) = z^q e^{a+bz} \prod_{\rho_P} \left(1 - \frac{z}{\rho_P}\right) e^{z/\rho_P},$$

where $\sum_{\rho_P} |\rho_P|^{-2}$ converges and that the infinite product is uniformly convergent on compacts sets of \mathbb{C} . The hypothesis that $P(z)$ does not vanish in $\Re(z) > r$ implies that

$$\left| \frac{P(r+it)}{P(r+\epsilon+it)} \right| \leq \left| \frac{r+it}{r+\epsilon+it} \right|^q \exp(\epsilon|\Re(b)| + \epsilon|\inf_{\rho_P} \Re(\rho_P)| \sum_{\rho_P} |\rho_P|^{-2}).$$

Next we restrict $\epsilon \in (0,1)$ if $r \geq 0$ and $\epsilon \in (0, |r|/2)$ otherwise and recall that $|\inf_{\rho_P} \Re(\rho_P)| < \infty$ holds for any Dirichlet polynomial $P(z)$ with zeros ρ_P (see (2.3)). Thus, there exist a positive constant C such that

$$\left| \frac{P(r+it)}{P(r+\epsilon+it)} \right| \leq C$$

when $\epsilon \rightarrow 0$. This estimate yields the desired upper bound

$$|H_\epsilon(s)| \leq \frac{C}{|s|},$$

which obviously belongs to $L^2(\Re(s) = 1/2)$. Hence, by Lebesgue's theorem $\mathcal{M}(\mathbf{1}_{(0,1)})$ belongs to the closure of $\mathcal{M}(C_r)$ in the $L^2(\Re(s) = 1/2)$ norm and this completes the proof.

4. THE ANALOG OF BURNOL'S LOWER BOUND FOR DIRICHLET POLYNOMIALS

Proof of Theorem 2. If the polynomial $P(s)$ does not have zeros on the line $\Re(s) = r$ the constant C must be zero, as in the statement of the theorem.

Suppose now that $P(s)$ possesses l distinct zeros $r+it_1, \dots, r+it_l$ on the line $\Re(s) = r$. By (2.5) and the fact that the Mellin transform is an isometry

$$d_{n,r}^2 = \inf_{A_n \in \mathcal{D}_n} \frac{1}{2\pi} \int_{\Re(s)=1/2} \left| \frac{1 - P(s+r-1/2)A_n(s)}{s} \right|^2 |ds|,$$

where \mathcal{D}_n is the space of Dirichlet polynomials of the form $\sum_{k=1}^n b_k k^{-s}$. Let us consider $Q_r(s) = P(s + r - 1/2)$ and the subspace $\mathcal{D}_{n,l} \subset \mathcal{D}_n$ composed by all Dirichlet polynomials $B_n(s)$ which satisfy the l interpolation conditions $B_n(1/2 + it_1) = 1, \dots, B_n(1/2 + it_l) = 1$. Since $1 - Q_r(s)A_n(s)$ is a DP of order mn and obeys the same interpolations property, then

$$d_{n,r}^2 \geq \inf_{B \in \mathcal{D}_{n,l}} \frac{1}{2\pi} \int_{\Re(s)=1/2} \left| \frac{B(s)}{s} \right|^2 |ds|.$$

Observe that $\mathcal{D}_{nm,l}$ is nonempty because it contains $B(s) \equiv 1$. If $B_{nm} \in \mathcal{D}_{nm,l}$ then $B_{nm}(1/2 + it_j) = 1$, $j = 1, \dots, l$ and $B_{nm}(1/2 + it) = \sum_{k=1}^{nm} b_k \psi_k(t)$, where ψ_k are defined in Section 2.4. The interpolation conditions can be rewritten in the form

$$A_{l,nm} \mathbf{B} = \mathbf{1}_l,$$

where

$$A = A_{l,nm} = \begin{pmatrix} \psi_1(t_1) & \psi_2(t_1) & \dots & \psi_{nm}(t_1) \\ \psi_1(t_2) & \psi_2(t_2) & \dots & \psi_{nm}(t_2) \\ \vdots & \vdots & & \vdots \\ \psi_1(t_l) & \psi_2(t_l) & \dots & \psi_{nm}(t_l) \end{pmatrix},$$

$\mathbf{B} = (b_1, \dots, b_{nm})^T$ and $\mathbf{1}_l$ is the column vector of size l whose entries are all equal to one. Then obviously

$$\|B_{nm}(1/2 + it)\|_{L^2(\mathbb{R}, \omega)}^2 = |b_1|^2 + |b_2|^2 + \dots + |b_{nm}|^2,$$

where $L^2(\mathbb{R}, \omega)$ is the weighted L^2 space with weight $\omega(t) = (1/2\pi)(1/4 + t^2)$. Thus the problem reduces to minimize $\|\mathbf{B}\|^2$, $\mathbf{B} \in \mathbb{C}^{nm}$, subject to $A_{l,nm} \mathbf{B} = \mathbf{1}_l$. The solution of this problem is known to be given by the solution of the following linear system (see [11, Theorem 2.19] and [19, 21]):

$$\begin{aligned} A \mathbf{B} &= \mathbf{1}_l, \\ \mathbf{B} &= A^* \lambda, \quad \lambda \in \mathbb{C}^l. \end{aligned}$$

By Lemma 3 there exists $n(l) \in \mathbb{N}$ such that the self-adjoint matrix $H = AA^* = (K_{nm}(t_i, t_j))_{i,j=1}^l$ is nonsingular for every $n > n(l)$. Hence, for $n > n(l)$, the above system has the unique solution $\tilde{\mathbf{B}} = A^* H^{-1} \mathbf{1}_l$. Since $AA^* = H$ and H is self-adjoint, then

$$\|\tilde{B}_{nm}(1/2 + it)\|_{L^2(\mathbb{R}, \omega)}^2 = \tilde{\mathbf{B}}^* \tilde{\mathbf{B}} = \mathbf{1}_l^* (H^{-1})^* A A^* H^{-1} \mathbf{1}_l = \mathbf{1}_l^* H^{-1} \mathbf{1}_l.$$

Thus, by the Cramer formula for the inverse matrix

$$\|\tilde{B}_{nm}(1/2 + it)\|_{L^2(\mathbb{R}, \omega)}^2 = \sum_{i,j=1}^l (-1)^{i+j} \frac{\det H_{ij}}{\det H},$$

where H_{ij} are the (i, j) -th cofactors of H . Moreover, the asymptotic relations (2.7) and (2.8) and Lemma 3 yield the following ones, as $n \rightarrow \infty$:

$$\begin{aligned} \det H_{jj} &\sim (\log nm)^{l-1} \frac{|1/2 + it_1|^2 \dots |1/2 + it_l|^2}{|1/2 + it_j|^2} \\ \det H_{ij} &= O((\log nm)^{l-2}), \quad i \neq j \\ \det H &\sim (\log nm)^l |1/2 + it_1|^2 \dots |1/2 + it_l|^2. \end{aligned}$$

Hence,

$$\|\tilde{B}_{nm}(1/2 + it)\|_{L^2(\mathbb{R}, \omega)}^2 \sim \frac{1}{\log n + \log m} \sum_{j=1}^l \frac{1}{1/4 + t_j^2}, \quad \text{as } n \rightarrow \infty,$$

Since $d_{n,r}^2 \geq \|\tilde{B}_{nm}(1/2 + it)\|_{L^2(\mathbb{R}, \omega)}^2$, we obtain

$$\liminf_{n \rightarrow \infty} d_{n,r}^2 \log n \geq \sum_{j=1}^l \frac{1}{1/4 + t_j^2}, \quad \text{as } n \rightarrow \infty.$$

Observe that the set of zeros $r + it_1, \dots, r + it_l$ was chosen to be arbitrary. Hence,

$$\liminf_{n \rightarrow \infty} d_{n,r}^2 \log n \geq C, \quad \text{as } n \rightarrow \infty,$$

where

$$C = \sum_{\Re(\rho_P)=r} \frac{1}{|\rho_P - r + 1/2|^2}.$$

□

Finally we comment the result of the theorem for the three possible situations for the location of the zeros of the Dirichlet polynomial.

1. When $P(s)$ possesses a zero ρ_P in the semi-plane $\Re(s) > r$, then we may write $\rho_P = r + \delta + it_0$ with $\delta > 0$. The function $f(x) := x^{-1/2+\delta-it_0}$ belongs to $L^2(0, 1)$ and by (2.5) we have

$$\int_0^1 \kappa_r \left(\frac{1}{kx} \right) \overline{f(x)} dx = \frac{P(r + \delta + it_0)}{1/2 + \delta + it_0} k^{-1/2-\delta-it_0} = 0.$$

Therefore f belongs to the orthogonal complement of the space $\text{span}\{\kappa(1/kx) : k = 1, \dots, n\}$ in $L^2(0, 1)$ for every $n \in \mathbb{N}$. Thus

$$d_{n,r}^2 \geq \frac{|\int_0^1 \mathbf{1}_{(0,1)}(x) \overline{f(x)} dx|^2}{\|f\|_{L^2(0,1)}^2} = \frac{2\delta}{|\rho_P - r + 1/2|^2},$$

that is, $\liminf_{n \rightarrow \infty} d_{n,r}^2 \log n = \infty$ and the lower bound proved in Theorem 2 is superficial.

2. When all the zeros of $P(s)$ belong to the semi-plane $\Re(s) \leq r$ but there is at least one zero on the line $\Re(s) = r$ the lower bound proved in the latter theorem is useful because Teorema 1 yields $\lim_{n \rightarrow \infty} d_{n,r}^2 = 0$ and Teorema 2 provides a lower bound for the speed of convergence of $d_{n,r}^2$. Moreover, employing a slightly modified argument from [9] we can show that the lower bound represents the true order of decrease of $d_{n,r}$, that is,

$$d_{n,r}^2 \sim \frac{C}{\log n},$$

for at least some specific Dirichlet polynomials.

3. When all the zeros of $P(s)$ belong to $\Re(s) < r$ the lower bound is trivial because $C = 0$. In this case there are two possibilities: either $\sup_{\rho_P} \{\Re(\rho_P)\} < r$ or $\sup_{\rho_P} \{\Re(\rho_P)\} = r$.

In the first one we may apply Lemma 2 with $f(s) = P(s + r - 1/2)$, $\sigma_z = \sup_{\rho_P} \{\Re(\rho_P)\} - r + 1/2$ and $\epsilon = r - \sup_{\rho_P} \{\Re(\rho_P)\}$ to obtain

$$\sum_{k=1}^n \frac{\mu_f(k)}{k^{1/2+it}} = \frac{1}{P(r + it)} + O_{r,\delta}(n^{-\delta})$$

for each $\delta < r - \sup_{\rho_P} \{\Re(\rho_P)\}$. As we have already mentioned in the proof of Theorem 2,

$$d_{n,r}^2 = \inf_{A_n \in \mathcal{D}_n} \frac{1}{2\pi} \int_{\Re(s)=1/2} \left| \frac{1 - P(s + r - 1/2)A_n(s)}{s} \right|^2 |ds|.$$

Then choosing $A_n(s) = \sum_{k=1}^n \mu_f(k)k^{-s}$ we obtain

$$d_{n,r}^2 \leq \frac{1}{2\pi} \int_{\mathbb{R}} \frac{|P(r + it)O_{r,\delta}(n^{-\delta})|^2}{1/4 + t^2} dt = O_{r,\delta}(n^{-\delta}).$$

Performing only a slight modification of the arguments it is possible to show that

$$d_{n,r}^2 \leq o(n^{-\delta})$$

for each $\delta < r - \sup_{\rho_P} \{\Re(\rho_P)\}$. On the other hand, using some estimates established in [20] and reasonings similar to those in the proof of Theorem 2, it can be shown that

$$d_{n,r}^2 \neq o(n^{-\delta})$$

for each $\delta > 2r - 2\sup_{\rho_P} \{\Re(\rho_P)\}$.

The second situation, that is when $\sup_{\rho_P} \{\Re(\rho_P)\} = r$, seems to be more complicated and we expect that a good lower bound for $d_{n,r}$ should depend not only on supremum of the real parts of the zeros of P but most probably on their further properties.

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DEPARTAMENTO DE MATEMÁTICA APLICADA, IBILCE, UNIVERSIDADE ESTADUAL PAULISTA, 15054-000 SÃO JOSÉ DO RIO PRETO, SP, BRAZIL.

E-mail address: `willian@ibilce.unesp.br`